



**national accelerator laboratory**

FN-232  
0402

INFINITE TRANSITION-ENERGY LATTICE  
USING  $\pi$ -STRAIGHT SECTIONS

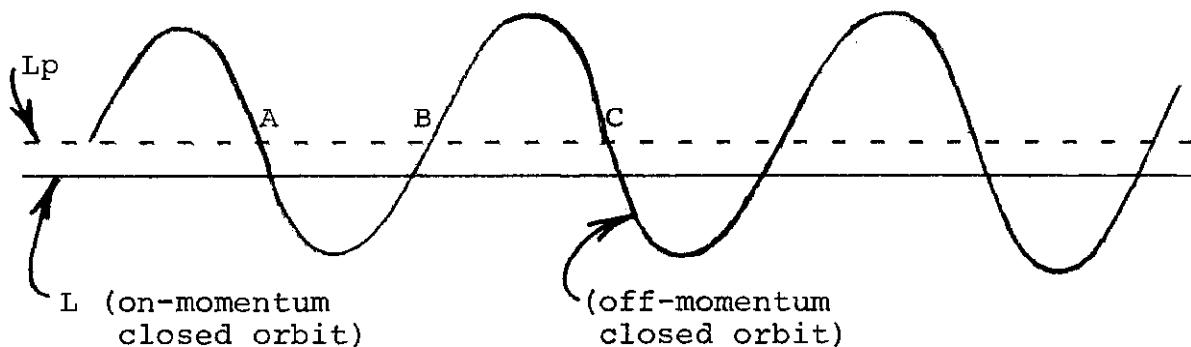
L. C. Teng

July 16, 1971

It is generally known that with reverse bending one can push  $\gamma_t$  (transition  $\gamma$ ) to  $\infty$  in a lattice (V.V. Vladimirski and E.K. Tarasov, "Theoretical Problems of the Ring Accelerators," Moscow, Academy of Sciences, 1955). We want to show here that the same can be accomplished by judicious insertion of the  $\pi$ -straight sections (A.A. Garren, p. 22, Proceedings of the V International Conference on High Energy Accelerators, Frascati, 1965). This effect was noticed, for example, in the lattice of the 500 MeV booster proposed for the Argonne ZGS (Argonne National Laboratory Accelerator Division Proposal, June 1969).



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I. GEOMETRICAL OBSERVATIONFigure 1

In an AG lattice if the on-momentum closed orbit is shown stretched out as straight line L (Fig. 1), the off-momentum orbit is oscillatory about line L. Because of the curvature of L (say, downward) an inward-swing section such as AB of the off-momentum orbit has a negative first-order length increment and an outward-swing section such as BC has a positive first-order length increment. The net orbit length increment is generally positive but rather small and corresponds roughly to that of the mean orbit shown as dotted straight line Lp. An AG lattice, therefore, generally has a small dispersion orbit-length increment, hence a large momentum compaction.

The concept of reducing (to zero) the dispersion orbit-length increment by introducing reverse bending is as follows: If the curvature of the on-momentum orbit L is reversed over the section BC the first-order orbit-length increment over BC will also reverse sign and become negative. The combined

orbit-length increment over AB and BC will, then, have a large excess negative value. By introducing a few such reverse bends it is possible to totally compensate the small net positive orbit-length increment corresponding to that of  $L_p$  and obtain a zero total dispersion orbit-length increment, hence  $\infty$  momentum compaction and infinite  $\gamma_t$ .

An interesting variance of this concept is the following: Instead of a reverse bend over BC one can make the orbit curvature zero, i.e., make section BC into a  $\pi$ -straight section. In this case the first-order orbit-length increment over BC is zero and the combined increment over AB and BC still has a large excess negative value although it is only roughly half that of the case with the reverse bend. Zero total dispersion orbit-length increment and infinite  $\gamma_t$  can similarly be attained. Compared to the case with reverse bending the  $\pi$ -insertion lattice has the following practical advantages:

- (a) No reverse bending magnet is required.
- (b) Long straight sections in a lattice are needed anyway for injection, extraction, rf cavities, etc.

It is clear from the discussion that any matched long straight section with a large phase advance close to  $\pi$  can be used effectively over section BC. But  $\pi$ -straight section is the simplest and can be analyzed most easily.

II. DISPERSION FORMALISM

We will write the  $3 \times 3$  horizontal optics-dispersion transfer matrix as

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} = (M|D) \quad (1)$$

where

$$\left\{ \begin{array}{l} M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2 \times 2 \text{ optics transfer matrix} \\ D = \begin{pmatrix} e \\ f \end{pmatrix} = \text{dispersion transfer vector.} \end{array} \right.$$

The following rules of multiplication are obvious:

(a) If a dispersion vector  $\begin{pmatrix} \xi \\ \xi' \\ 1 \end{pmatrix}$  is written as

$$E = \begin{pmatrix} \xi \\ \xi' \end{pmatrix}$$

then

$$(M|D)E = ME + D. \quad (2)$$

$$(b) (M_2|D_2)(M_1|D_1) = (M_2M_1|M_2D_1+D_2) \quad (3)$$

$$(c) (M|D)^n = (M^n|(1-M^n)(1-M)^{-1}D). \quad (4)$$

(d) If  $(M|D)$  is across a periodicity cell and  $x = \begin{pmatrix} x \\ x' \end{pmatrix}$  is the periodic dispersion vector (this is conventionally written as  $\begin{pmatrix} x_p \\ x'_p \end{pmatrix}$ ), but since in this paper we will deal

only with the dispersion vector and never with the free oscillation vector we can, without confusion, drop the subscript p), then

$$(M|D)X = MX + D = X$$

and

$$D = (1-M)X. \quad (5)$$

- (e) For the  $\pi$ -straight section the optics-dispersion transfer matrix is  $(-1|0)$ .

### III. EFFECT OF $\pi$ -STRAIGHT SECTION INSERTION

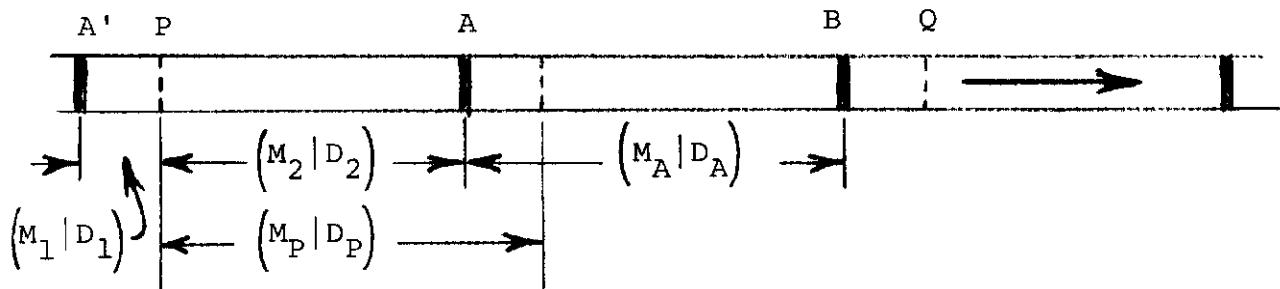


Figure 2

We start with a lattice without  $\pi$ -insertion (Figure 2). The distance between A and B is one repetition length which may contain many cells. We need the following relations between the optics-dispersion transfer matrices defined in Fig. 2. From A to B we have

$$(M_A | D_A) = (M_2 | D_2) (M_1 | D_1) = (M_2 M_1 | M_2 D_1 + D_2)$$

which gives

$$M_2^{D_1+D_2} = D_A = (1-M_A) X_A . \quad (6)$$

From P to Q we have

$$\begin{aligned} (M_P | D_P)^2 &= (M_1 | D_1)(M_2 | D_2)(M_1 | D_1)(M_2 | D_2) \\ &= (M_1 M_2 M_1 M_2 | M_1 M_2 M_1 D_2 + M_1 M_2 D_1 + M_1 D_2 + D_1) \end{aligned}$$

which gives

$$M_1 M_2 M_1 D_2 + M_1 M_2 D_1 + M_1 D_2 + D_1 = (1 - M_P^2) X_P . \quad (7)$$

Now we insert  $\pi$ -straight sections at A, B, and all other homologous locations. The new transfer matrix from P to Q becomes

$$\begin{aligned} &(M_1 | D_1) (-1 | 0) (M_2 | D_2)(M_1 | D_1) (-1 | 0) (M_2 | D_2) \\ &= (M_1 M_2 M_1 M_2 | M_1 M_2 M_1 D_2 - M_1 M_2 D_1 - M_1 D_2 + D_1) . \end{aligned}$$

Together with Eqs. (6) and (7) this gives for the new periodic dispersion vector at P,  $\bar{X}_P = \begin{pmatrix} \bar{x} \\ \bar{x}' \end{pmatrix}_P$ ,

$$\begin{aligned} (1 - M_P^2) \bar{X}_P &= M_1 M_2 M_1 D_2 - M_1 M_2 D_1 - M_1 D_2 + D_1 \\ &= M_1 M_2 M_1 D_2 + M_1 M_2 D_1 + M_1 D_2 + D_1 - 2M_1 (M_2 D_1 + D_2) \\ &= (1 - M_P^2) X_P - 2M_1 (1 - M_A) X_A \end{aligned} \quad (8)$$

whence, we get

$$\Delta X_P \equiv \bar{X}_P - X_P = -2 (1 - M_P^2)^{-1} M_1 (1 - M_A) X_A . \quad (9)$$

To simplify the right-hand-side expression we note first

$$\left(1-M_P^{-2}\right)^{-1} = \frac{1-M_P^{-2}}{4\sin^2 \mu}$$

where  $\mu$  is the phase advance across one repetition length (not including the  $\pi$ -insertion). Furthermore,

$$\begin{aligned} \left(1-M_P^{-2}\right)M_1 &= \left(1-M_2^{-1}M_1^{-1}M_2^{-1}M_1^{-1}\right)M_1 \\ &= M_1 \left(1-M_1^{-1}M_2^{-1}M_1^{-1}M_2^{-1}\right) = M_1 \left(1-M_A^{-2}\right) \end{aligned}$$

and

$$\begin{aligned} \left(1-M_A^{-2}\right) \left(1-M_A\right) &= \left(2-M_A-M_A^{-1}\right) \left(1+M_A^{-1}\right) \\ &= 2(1-\cos\mu) \left(M_A^{1/2} + M_A^{-1/2}\right) M_A^{-1/2} \\ &= 4\cos\frac{\mu}{2}(1-\cos\mu) M_A^{-1/2} \end{aligned}$$

where  $M_A^{-1/2}$  is the same as  $M_A$  except with the phase advance  $\mu$  replaced by  $-\frac{\mu}{2}$ . Altogether we have

$$\begin{aligned} \Delta X_P &= -2 \left(1-M_P^{-2}\right)^{-1} M_1 \left(1-M_A\right) X_A \\ &= -\frac{1}{2\sin^2 \mu} M_1 \left(1-M_A^{-2}\right) \left(1-M_A\right) X_A \\ &= -\frac{1}{\cos\frac{\mu}{2}} M_1 M_A^{-1/2} X_A. \end{aligned} \tag{10}$$

$M_1$  is the matrix from the beginning of the repetition length (point A' in Fig. 1) to P and by definition  $M_A^{-1/2}$  reduces the

phase advance in  $M_1$  by  $\frac{\mu}{2}$ . Therefore,  $M_1 M_A^{-1/2}$  has the form

$$M_1 M_A^{-\frac{1}{2}} = \begin{pmatrix} \sqrt{\beta_P} (\cos \phi_P + \alpha_A \sin \phi_P) & \sqrt{\beta_A \beta_P} \sin \phi_P \\ -\sqrt{\beta_A \beta_P} \left[ (\alpha_P - \alpha_A) \cos \phi_P + (1 + \alpha_A \alpha_P) \sin \phi_P \right] & \sqrt{\beta_A} (\cos \phi_P - \alpha_P \sin \phi_P) \end{pmatrix} \quad (11)$$

where

$$\phi_P = (\text{phase advance from } A' \text{ to } P) - \frac{\mu}{2}$$

and where all betatron oscillation functions  $\alpha_A$ ,  $\beta_A$ ,  $\alpha_P$ ,  $\beta_P$  and the phases are values before the insertion of the  $\pi$ -straight sections. Of course, Eq. (11) can be derived also by straightforward computation. We are only interested in the first row  $\Delta x_P$  of the vector Eq. (10) and since  $P$  is the running variable point within the repetition length we can drop the subscript  $P$ . This gives

$$\Delta x = \bar{x} - x = - \frac{\sqrt{\beta}}{\cos \frac{\mu}{2}} \left[ (\cos \phi + \alpha_A \sin \phi) \sqrt{\beta_A} x_A' + \sin \phi \sqrt{\beta_A} x_A \right] \quad (12)$$

with

$$\begin{aligned} \phi &= (\text{phase measured from beginning of repetition length}) - \frac{\mu}{2} \\ &= (\text{phase measured from "middle" of repetition length}). \end{aligned}$$

Eq. (12) can be derived in different ways, e.g., by matching the solutions of the orbit equations of the orbit equation with inversions at points  $A$ ,  $B$ , etc. to form the closed orbit. The derivation given above seems to be the shortest and most lucid.

The new dispersion orbit-length increment per repetition length with the  $\pi$ -insertions is (The orbit-length increment in

the  $\pi$ -straight section is zero.)

$$\begin{aligned} \int_{-\mu/2}^{\mu/2} \frac{x}{\rho} \beta d\phi &= \int_{-\mu/2}^{\mu/2} \frac{x}{\rho} \beta d\phi + \int_{-\mu/2}^{\mu/2} \frac{\Delta x}{\rho} \beta d\phi \\ &= \int_{-\mu/2}^{\mu/2} \frac{x}{\rho} \beta d\phi - \frac{1}{\cos \frac{\mu}{2}} \left[ (C + \alpha_A S) \sqrt{\frac{x_A}{\beta_A}} + S \sqrt{\beta_A} x'_A \right] \end{aligned} \quad (13)$$

where

$$\left\{ \begin{array}{l} C \equiv \int_{-\mu/2}^{\mu/2} \frac{\beta^{3/2}}{\rho} \cos \phi d\phi \\ S \equiv \int_{-\mu/2}^{\mu/2} \frac{\beta^{3/2}}{\rho} \sin \phi d\phi \end{array} \right.$$

and  $\rho = \rho(\phi)$  is the radius of curvature of the on-momentum orbit. Now we see that because of the denominator  $\cos \frac{\mu}{2}$ , by judicious choice of  $\mu$  we can adjust the second term on the right-hand-side of Eq. (13) to cancel the first term to give zero dispersion orbit-length increment or infinite  $\gamma_t$ .

#### IV. CRUDE ESTIMATE AND EXAMPLES

To give a simple example let us assume that the  $\pi$ -straight sections are inserted at locations where  $\alpha_A = 0$ , hence  $x'_A = 0$ . The condition for zero dispersion orbit-length increment Eq. (13) then gives

$$\int_{-\mu/2}^{\mu/2} \frac{x}{\rho} \beta d\phi = \frac{C}{\cos \frac{\mu}{2}} \sqrt{\beta_A} \quad (14)$$

Now, for a very crude estimate we have

$$C = \int_{-\mu/2}^{\mu/2} \frac{\beta^{3/2}}{\rho} \cos \phi d\phi = \left\langle \frac{\beta^{3/2}}{\rho} \right\rangle \int_{-\mu/2}^{\mu/2} \cos \phi d\phi = 2 \left\langle \frac{\beta^{3/2}}{\rho} \right\rangle \sin \frac{\mu}{2} \quad (15)$$

where  $\langle \rangle$  denotes some kind of average value in the periodicity length from  $-\frac{\mu}{2}$  to  $\frac{\mu}{2}$ . Also

$$\int_{-\mu/2}^{\mu/2} \frac{x}{\rho} \beta d\phi = \left\langle \frac{\beta^{3/2}}{\rho} \right\rangle \left\langle \frac{x}{\sqrt{\beta}} \right\rangle \int_{-\mu/2}^{\mu/2} d\phi = \left\langle \frac{\beta^{3/2}}{\rho} \right\rangle \left\langle \frac{x}{\sqrt{\beta}} \right\rangle \mu \quad (16)$$

where  $\langle \rangle$  again denotes some kind of average. Assuming  $\left\langle \frac{\beta^{3/2}}{\rho} \right\rangle$  in Eq. (15) to be roughly equal to that in Eq. (16) and since  $\frac{x}{\sqrt{\beta}}$  is more or less constant in a normal lattice, hence  $\left\langle \frac{x}{\sqrt{\beta}} \right\rangle \approx \frac{x_A}{\sqrt{\beta_A}}$  we obtain by substituting Eqs. (15) and (16) in Eq. (14)

$$\frac{\mu}{2} \approx \tan \frac{\mu}{2}. \quad (17)$$

Let

$$\mu = (2n-1)\pi - 2\pi\varepsilon \quad n = \text{integer.}$$

We have

$$\tan \frac{\mu}{2} = \cot \varepsilon\pi$$

and

$$2\pi\nu = N(\mu + \pi) = 2\pi nN - 2\pi\varepsilon N$$

or

$$\nu = nN - \varepsilon N \quad (18)$$

where  $N$  is the number of symmetrically inserted  $\pi$ -straight sections and  $v$  is the betatron oscillation wave number including the  $\pi$ -insertions. In the first place  $v$  must have a value (integer)  $\pm \frac{1}{4}$ . Setting  $\epsilon N = \frac{1}{4}$  we get

$$\epsilon = \frac{1}{4N} \quad (19)$$

and that the integral part of  $v$  must be a multiple of  $N$ . Solving Eq. (18) for  $n$  in terms of  $v$  and inserting into  $\mu$  we get

$$\mu = 2\pi \left( \frac{v}{N} - \frac{1}{2} \right). \quad (20)$$

Substituting Eqs. (19) and (20) in Eq. (17) we get

$$v \approx N \left( \frac{1}{2} + \frac{1}{\pi} \cot \frac{\pi}{4N} \right). \quad (21)$$

This equation gives an approximate upper limit of  $v$  for a given number  $N$  of  $\pi$ -insertions. For low values of  $N$  we have

<u>N</u>	<u>v</u>	<u>N</u>	<u>v</u>
1	0.82	6	17.5
2	2.54	7	23.3
3	5.06	8	29.9
4	8.40	9	37.2
5	12.5	10	45.4

Suppose we use FODO cells similar to those of the NAL main ring and insert 6 ( $N = 6$ )  $\pi$ -straight sections to build up an infinite- $\gamma_t$  lattice for 400 BeV and  $v \approx 18 - \frac{1}{4}$  ( $n = 3$ ). There are 15 normal cells in between two  $\pi$ -insertions. At 400 BeV the orbit parameters obtained from computer runs using SYNCH are

orbit radius = 988.0 m

bending magnet field = 19.225 kG

quadrupole field gradient =  $\pm 213.0$  kG/m

$v_x = 17.772$

$v_y = 17.818$

$$\gamma_t^2 = \begin{cases} -1811 & \text{mid-D insertion} \\ -4778 & \text{mid-F insertion} \end{cases}$$

Other parameters are given in the SYNCH output attached. Instead of  $\gamma_t$  which may be imaginary we print out  $\gamma_t^2$ . The two values of  $\gamma_t^2$  correspond to the cases in which the  $\pi$ -straight sections are inserted in the middle of either the defocusing or the focusing quadrupoles. The large negative  $\gamma_t^2$  values (instead of  $\infty$ ) indicate that we have overshot in reducing the dispersion orbit-length increment and the increment is actually slightly negative in both cases (negative momentum compaction). This is presumably because the  $v_x$  values chosen are closer to 18 than  $\frac{1}{4}$ .

The  $\beta$ -values are rather large in the  $\pi$ -straight sections which is inherent in the design of the  $\pi$ -straight section. A serious drawback of these lattices is the very large local maximum  $x$  ( $x_p$  or  $x_{eq}$ ) which makes very stringent demand on the magnet aperture at these locations. This is, of course, inherent in the method employed to compensate for the dispersion orbit-length increase. These large local maximum  $x$  can be reduced by increasing the number of  $\pi$ -insertions (say, from 6 to 9).

The modifications of SYNCH to print out  $\gamma_t^2$  was made by Dr. J. MacLachlan. His assistance is gratefully acknowledged.

## SYNCH RUN MR01

MAIN DING WITH NEW L S INSPECTION FOR HIGH GAMMA-I

## MID-D INSERTION

***	EE		213.
***	GD		/ /
***	GRHO		/ /
***	GR		/ /
***	MAC		/ /
***	GF		/ /
***	MAS		/ /
***	GD		/ /
***	DRF		/ /
***	DRF		/ /
***	DRF		/ /
***	DRF		/ /

## NORMAL CELL

CYC 22 // 00 00

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

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B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

B 0 0 R 0 A C R 0

## BETATRON FUNCTIONS THROUGH C

PSIX RETAX ALOHAX XFO

DXFO BETAX

PSIY

RETAY ALPHAY

YEQ

DYEQ

DET

DF

-13-

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\*16456660

THETA = .75981117 QX =

\*16413848 QY =

GAMMA-1, SQUARE = 20.427153701

MAXIMA BETAX 111= 102.90832138 XE01 111= 5.31269263 BETAY1 111= 102.62603720

EXPLORATION OF THE CLOUD

MID-D INSERTION

PERFECTATION FUNCTIONS THROUGH  $\zeta$  PSIX BELL

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1	103-228570	0F	.37328	165.9514	6.064	42.08611	-1.76978	12.58221	+43.2534	33.01132	-6.116	0.00000	0.00000	5.-3
16	106-5118872	0F	.37446	150.4376	6.391	40.0096	-2.48888	12.26530	+43.758	40.7536	-6.995	0.00000	0.00000	3.1
17	106-8486071	000	.34028	47.4693	6.365	29.38139	-2.48888	9.35251	+44.920	83.9007	-5.995	0.00000	0.00000	1.-7
18	109-952282	00	.38237	75.8621	4.581	26.98383	-2.010178	8.70988	+45.109	95.5243	-4.835	0.00000	0.00000	1.-7
19	111-01967	00	.38474	67.6849	3.130	25.07723	-1.56317	6.22709	+45.279	104.2832	-3.326	0.00000	0.00000	1.-7
20	112-228567	00	.38717	62.3417	1.908	23.62579	-1.15605	7.89558	+45.337	109.5462	-1.578	0.00000	0.00000	17.-4
21	113-151327	00	.39016	59.4425	.826	22.60123	-7.76391	7.70391	+45.591	110.9341	-2.85	0.00000	0.00000	11.-1
22	142-33662	0.0000	.39050	35.1506	.000	.30970	-7.76391	5.94564	+50.000	102.95251	-0.000	0.00000	0.00000	11.-1
23	201-870342	C	.66414	35.3506	.000	-21.67005	-32.097	5.94564	+66.665	102.6260	.000	0.00000	0.00000	11.-1
24	261-317222	C	.68828	35.1506	.000	-19.42367	-4.55118	5.94564	+82.323	102.6260	.000	0.00000	0.00000	11.-1

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25	320.79702	C	-99492	35.1506	-0.00	4.863198	-76908	5.34564	-99394
26	380.28182	C	-115555	35.1506	-0.00	27.1E492	-35599	5.94564	1.18559
27	439.7062	C	1.12063	35.1506	-0.00	26.13444	-5.02323	5.94564	1.12123
28	499.25742	C	1.48043	35.1506	-0.00	2.895643	-76811	5.44544	1.788
29	558.744922	C	1.64697	35.1506	-0.00	-20.27461	-58601	5.94554	1.65253

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30.	618.23102.	C	1.81111	35.1505	-.000	-20.58223	-37275	5.94554	1.81117	102.6260	.000	0.00000	0.00000	11.1
31.	677.1782	C	1.97125	35.1506	-.000	2.27278	67639	5.77654	1.81182	102.6260	.000	0.00000	0.00000	11.1
32.	727.20462	C	2.14104	35.1506	-.000	26.06132	41934	5.94564	2.14547	102.6260	.000	0.00000	0.00000	11.1
33.	796.69192	C	2.37152	35.1506	-.000	27.6633	-33814	5.94554	2.31111	102.6260	.000	0.00000	0.00010	11.1
34.	856.17822	C	2.46956	35.1505	-.000	5.46622	-76372	5.94564	2.47576	102.6260	.000	0.00000	0.00000	11.1

35	915.66502	C	2.63310	35.3595	-.000	-18.86580	-.44753	5.34554	2.57741	102.6260	.000	0.00000	0.00030	13.1
36	975.5182	C	2.79794	35.3505	-.000	-21.72462	-.30895	5.94564	2.83205	102.6260	.000	0.00000	0.00030	13.1
37	1034.61862	C	2.95201	35.3505	-.000	-.30910	.76391	5.94564	2.96910	102.6260	.000	0.00000	0.00030	13.1

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THEIA 5 6-2831 8522 8X 11-11246129 101 17-81813445

GAMMA-1 SQUARE = -18.11007E+02

REIAMI-1112-179-96062717 XE01 1301= 44.513301926 BEIAV( 9)= 281.45094544



